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LETTER TO THE EDITOR

On the structure of eigenvectors of the multidimensional Lamé operator

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Abstract. We describe the finite-dimensional functional manifold which contains all meromorphic solutions to the multidimensional generalization of the Lamé equation.

In this letter we consider the long standing problem of finding the way to the explicit construction of the eigenvectors of the Schrödinger operator

$$H_{M,n} = -\frac{1}{2}\sum_{j=1}^{M}\left(\frac{\partial}{\partial x_j}\right)^2 + n(n+1)\sum_{j>l}^{M}\wp(x_j - x_l) \qquad n \in \mathbb{Z}_+ \tag{1}$$

where $\wp(x)$ is the Weierstrass elliptic function with two periods ω_1 and ω_2, $\operatorname{Im} m(\omega_2\omega_1^{-1}) > 0$. In physical applications where $H_{M,n}$ has to be Hermitian, these periods are usually chosen such that $\omega_1 \in \mathbb{R}_+$, $\mathrm{i}^{-1}\omega_2 \in \mathbb{R}_+$.

The operator (1) describes the quantum system of M particles interacting via two-body potential $n(n+1)\wp(x)$. The analogous classical many-particle problem has been proven to be completely integrable in the Liouville sense [1, 2]. The construction of the set of corresponding quantal 'integrals of motion' $\{I_l\}(1 \le l \le M-1)$ commuting with $H_{M,n}$ has also been performed by using the quantum analogue of the classical Lax representation [3]. Moreover, the superintegrability, i.e. the existence of more than $M-1$ functionally independent operators commuting with $H_{M,n}$ has been conjectured [4] and proved in the simplest case $M = 2$ in which the corresponding eigenproblem reduces to the usual Lamé equation and the structure of eigenfunctions was described more than a century ago [5]. Despite many results [3, 4, 6, 7] obtained for trigonometric and rational degenerations of (1), the explicit form of any solution to $H_{M,n}\psi = E\psi$ for $M \ge 3$ in general elliptic case has still not been indicated. Up to now it has not been clear how one should use the rather complicated operators $\{I_l\}$ containing higher derivatives for reduction of the spectral problem.

Our observation consists in using the simpler symmetry of $H_{M,n}$ for this purpose. Since $\wp(x)$ is double periodic, it is easy to see that (1) commutes with the $2M$ shift operators $Q_{\alpha j} = \exp(\omega_\alpha \partial/\partial x_j)$, $\alpha = 1, 2$, $1 \le j \le M$. Let $\psi^{(q)}(x_1, \ldots, x_M)$ be their common eigenvector,

$$\psi^{(q)}\left(x_1 + \sum_{\alpha=1}^{2} l_1^{(\alpha)}\omega_\alpha, \ldots, x_M + \sum_{\alpha=1}^{2} l_M^{(\alpha)}\omega_\alpha\right) = \exp\left(\mathrm{i}\sum_{j=1}^{M}\sum_{\alpha=1}^{2} q_\alpha^{(j)} l_j^{(\alpha)}\right)\psi^{(q)}(x_1, \ldots, x_M) \tag{2}$$

where $l_j^{(\alpha)} \in \mathbb{Z}$ and $q_\alpha^{(j)} \in \mathbb{C} (\mathrm{mod}\, 2\pi)$. Hence $\psi^{(q)}(x_1, \ldots, x_M)$ can be treated on the M-dimensional torus $\mathbb{T}^M = (\mathbb{C}/\Gamma)^M$, $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with quasiperiodic boundary conditions (2). The structure of singularities of $H_{M,n}$ on this torus shows that $\psi^{(q)}$ is analytic on $\mathbb{T}^M \backslash \mathbb{P}_M$, $\mathbb{P}_M$ being the set which consists of all $M(M-1)/2$ hypersurfaces P_{jk} defined by the equalities $x_j = x_k$, $1 \le j < k \le M$. On each P_{jk}, $\psi^{(q)}$ has a pole of nth order

$$\psi^{(q)}(x_1, \ldots, x_M)|_{x_j \to x_k} \sim (x_j - x_k)^{-n} \phi^{(jk)}(x_1, \ldots, x_M) \tag{3}$$

where $\phi^{(jk)}$ do not contain any singularities as $x_j \to x_k$. Let $\Psi_{M,n}$ be the class of functions analytic on $\mathbb{T}^M \backslash \mathbb{P}_M$ satisfying the relations (2) and (3).

Our main result consists in combining these properties so as to reduce the problem of finding $\psi^{(q)}(x_1, \ldots, x_M)$ to an algebraic one.

Proposition 1. The class $\Psi_{M,n}$ is a functional manifold of dimension $2M - 1 + n(nM)^{M-2}$. The parameters $\{q_\alpha^{(j)}\}$ are not independent but connected by the linear relation

$$\sum_{j=1}^{M} \left(q_1^{(j)} \omega_2 - q_2^{(j)} \omega_1 \right) \in 2\pi\Gamma . \tag{4}$$

The manifold $\Psi_{M,n}$ can be described as a union of the $(2M-1)$-parametric family of linear spaces $L(q)$, $\dim L(q) = n(nM)^{M-2}$, with the basis vectors parametrized by $q = \{q_\alpha^{(j)}\}$ satisfying (4).

The scheme of the proof is based on investigating the entire functions arising after explicit representation of the leading singularities of $\psi^{(q)}(x_1, \ldots, x_M)$ in terms of inverse powers of the Weierstrass sigma function $\sigma(x)$ given as a function of $\wp(x)$ by the relations $\mathrm{d}^2(\log \sigma(x))/\mathrm{d}x^2 = -\wp(x)$, $\lim_{x\to 0} x^{-1}\sigma(x) = 1$.

Proposition 2. The co-ordinate system on $\Psi_{M,n}$ can be chosen in such a way that all its elements are expressed through the Riemann theta functions of genus 1.

The first part of the proof consists in the explicit representation of the basis vectors of $L(q)$ in terms of the $(M-1)$-dimensional theta functions with the Riemannian B matrix having all diagonal and all non-diagonal elements equal, $B_{jk} = (\pi \mathrm{i} \omega_2 / nM\omega_1)(1 + \delta_{jk})$, $1 \le j, k \le M-1$. The second part is simply an application of the Appell reduction theorem [8].

Let $\psi_{M,n}$ be the submanifold of $\Psi_{M,n}$ which consists of the eigenvectors of $H_{M,n}$. The elements of $\psi_{M,n}$ are determined by $M+1$ complex parameters which include the $M-1$ relative particle quasimomenta, the total momentum and the trivial normalization factor. So the problem of selecting $\psi_{M,n}$ from $\Psi_{M,n}$ is equivalent to finding the solutions to $M - 2 + n(nM)^{M-2}$ purely algebraic equations which arise under substitution of the general expression for the elements of $\Psi_{M,n}$ into the eigenequation $H_{M,n}\psi = E\psi$.

The explicit realization of this procedure for the simplest non-trivial example $M = 3, n = 1$ is as follows. The appropriate parametrization of the elements of $\Psi_{3,1}$ can be written as

$$\psi(x_1, x_2, x_3) = A \left[\prod_{j>k=1}^{3} \sigma(x_j - x_k) \right]^{-1} \exp\left(\mathrm{i} \sum_{j=1}^{3} q_j x_j \right) \left(B + \frac{\partial}{\partial \gamma_{12}} + \frac{\partial}{\partial \gamma_{23}} + \frac{\partial}{\partial \gamma_{31}} \right)$$
$$\times\, \sigma(x_1 - x_2 + \gamma_{12}) \sigma(x_2 - x_3 + \gamma_{23}) \sigma(x_3 - x_1 + \gamma_{31}). \tag{5}$$

To describe $\psi_{3,1}$, one has to find four constraints to the parameters $B, \{q\}, \{\gamma\}$ in (5). The calculations may be carried out more easily with the use of the following extended version of the Liouville theorem for double quasiperiodic functions.

Proposition 3. Let $\varphi(x)$ be analytic on $\mathbb{T} = \mathbb{C}/\Gamma$ and $\varphi(x+\omega_1) = \exp(\mathrm{i}\theta_1)\varphi(x)$, $\varphi(x+\omega_2) = \exp(\mathrm{i}\theta_2)\varphi(x)$. Then $\varphi(x) \equiv 0$ if $\theta_1\omega_2 - \theta_2\omega_1 \notin 2\pi\Gamma$.

Due to this statement it is sufficient to choose the parameters in (5) so as to remove all the singularities of $\varphi(x_1, x_2, x_3) = \big(H_{3,1} - E\big)\psi(x_1, x_2, x_3)$ treated as the function of x_1 at fixed values of two other arguments. One gets the constraints defining $\psi_{3,1}$ in the form

$$B = \tfrac{1}{2}[\zeta(\gamma_{12}) + \zeta(\gamma_{23}) + \zeta(\gamma_{31}) - 3\zeta(\gamma_{12}+\gamma_{23}+\gamma_{31})] \tag{6}$$

$$\begin{aligned}
\mathrm{i}(q_1 - q_2) &= \tfrac{1}{2}[\zeta(\gamma_{23}) + \zeta(\gamma_{31}) - \zeta(\gamma_{12}) + 4\zeta(\gamma_{23}+\gamma_{31}) - 3\zeta(\gamma_{12}+\gamma_{23}+\gamma_{31})] \\
\mathrm{i}(q_2 - q_3) &= \tfrac{1}{2}[\zeta(\gamma_{31}) + \zeta(\gamma_{12}) - \zeta(\gamma_{23}) + 4\zeta(\gamma_{31}+\gamma_{12}) - 3\zeta(\gamma_{12}+\gamma_{23}+\gamma_{31})] \\
&\quad \times\zeta(\gamma_{12}) + \zeta(\gamma_{23}) + \zeta(\gamma_{31}) - 9\zeta(\gamma_{12}+\gamma_{23}+\gamma_{31}) \\
&\quad +4[\zeta(\gamma_{12}+\gamma_{23}) + \zeta(\gamma_{23}+\gamma_{31}) + \zeta(\gamma_{31}+\gamma_{12})] = 0
\end{aligned} \tag{7}$$

where $\zeta(x) = \mathrm{d}(\log\sigma(x))/\mathrm{d}x$ is the Weierstrass zeta function. The corresponding eigenvalue of $H_{3,1}$ is

$$\begin{aligned}
E = \tfrac{1}{2}\left(q_1^2 + q_2^2 + q_3^2\right) &- \tfrac{1}{2}[\zeta^2(\gamma_{12}) + \zeta^2(\gamma_{23}) + \zeta^2(\gamma_{31})] \\
&+ \tfrac{3}{2}[\zeta(\gamma_{12}) + \zeta(\gamma_{23}) + \zeta(\gamma_{31})]\zeta(\gamma_{12}+\gamma_{23}+\gamma_{31}) \\
&+ 2[\varrho(\gamma_{12}) + \varrho(\gamma_{23}) + \varrho(\gamma_{31})] \\
&- 2[\zeta(\gamma_{31})\zeta(\gamma_{12}+\gamma_{23}) + \zeta(\gamma_{23})\zeta(\gamma_{12}+\gamma_{31}) + \zeta(\gamma_{12})\zeta(\gamma_{23}+\gamma_{31})] \\
&- \tfrac{9}{4}[\zeta(\gamma_{12}) + \zeta(\gamma_{23}) + \zeta(\gamma_{31}) - \zeta(\gamma_{12}+\gamma_{23}+\gamma_{31})]^2 .
\end{aligned} \tag{8}$$

Since the relations (6)–(8) are invariant under cyclic permutations of the indices of $\{q\}$ and $\{\gamma\}$, the complete symmetrization of (5) on x_1, x_2, x_3 also gives the eigenfunctions of $H_{3,1}$ if $\{q\}$ and $\{\gamma\}$ obey (6), (7).These eigenfunctions are regular as $x_j - x_k \to 0$. The discrete spectrum of the corresponding three-particle system on the real circle $x_{1,2,3} \in \mathbb{R} \bmod \omega_1$ in the centre-of-mass frame ($q_1 + q_2 + q_3 = 0$) can be obtained by imposing the periodic boundary conditions

$$\left.\begin{aligned}
(q_1 - q_2)\omega_1 - 2\mathrm{i}\zeta\,(\omega_1/2)\,(2\gamma_{12} - \gamma_{23} - \gamma_{31}) &= 2\pi l_1 \\
(q_2 - q_3)\omega_1 - 2\mathrm{i}\zeta\,(\omega_1/2)\,(2\gamma_{23} - \gamma_{12} - \gamma_{31}) &= 2\pi l_2
\end{aligned}\right\} \quad l_1, l_2 \in \mathbb{Z}.$$

The extension of this procedure to the cases $M > 3$ and $n > 1$ which are more complex from a technical viewpoint, is under study and will be published elsewhere.

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References

[1] Calogero F 1975 *Lett. Nuovo Cimento* **13** 411
[2] Moser J 1975 *Adv. Math.* **16** 1

[3] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
[4] Chalykh O A and Veselov A P 1990 *Commun. Math. Phys.* **126** 597
[5] Halphen G H 1888 *Fonctions Elliptiques II* (Paris)
[6] Sutherland B 1971 *Phys. Rev.* A **4** 2019; 1972 *Phys. Rev.* A **5** 1378
[7] Sekiguchi J 1977 *Publ. RIMS Kyoto Univ.* **12** 455
[8] Appell P 1882 *Bull. Soc. Math. France* **10** 59